THE UNIVERSITY OF TEXAS AT AUSTIN

Greedy Subspace Clustering

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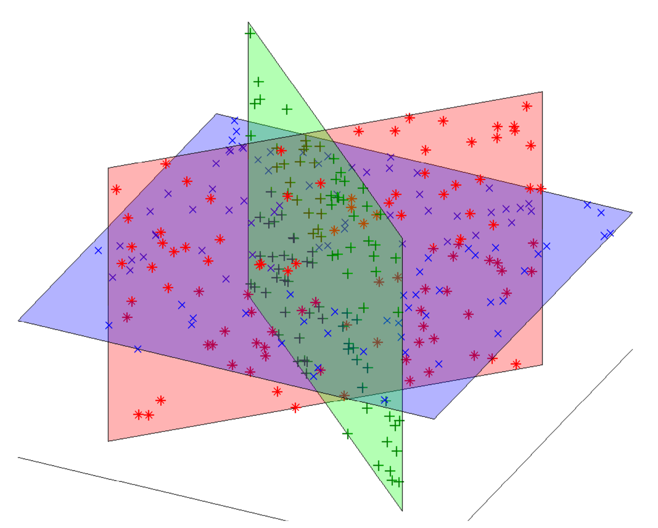
Electrical and Computer Engineering

Subspace Clustering



- Approximating data points with unions of low-dimensional subspaces
- Unions of subspaces model: Mixed datasets with latent labels, each label representing a linear subspace model.
- Task: Jointly find subspaces and cluster the points near each subspace

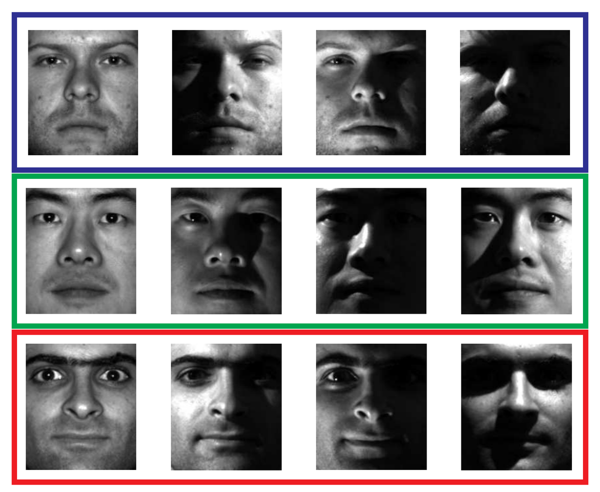
Motion Segmentation

- Feature points of a moving rigid object lie on a low-dimensional subspace

(Ref.: Hopkins155 dataset)

Face Clustering

- Face images of a person under varying illumination conditions lie on a low-dimensional subspace



(Ref.: Extended Yale B dataset)

Our contribution

Simple algorithms with

- Provable exact clustering guarantee without any subspace conditions
- Competitive performance and fast running time in practice

Related work & Comparison

Algorithms with theoretical guarantees involve two steps:
1. Neighborhood construction 2. Clustering/Subspace recovery

Algorithm	Neighborhood construction	Clustering	Sufficient conditions for: Fully random model	Semi-random model
SSC [1,2]	ℓ_1-minimization	Spectral	$\frac{d}{p} = O(\frac{\log(n/d)}{\log(nL)})$	$\max \text{aff} = O(\frac{\sqrt{\log(n/d)}}{\log(nL)})$
LRR [3]	Nuclear norm min.	Spectral	-	-
SSC-OMP [4]	OMP	Spectral	-	-
TSC [5]	Thersholding	Spectral	$\frac{d}{p} = O(\frac{1}{\log(nL)})$	$\max \text{aff} = O(\frac{1}{\log(nL)})$
LRSSC [6]	Hybrid $\ell_1 + \|\cdot\|_*$	Spectral	$\frac{d}{p} = O(\frac{1}{\log(nL)})$	-
NSN+GSR	NSN	GSR	$\frac{d}{p} = O(\frac{\log n}{\log(ndL)})$	$\max \text{aff} = O(\sqrt{\frac{\log n}{(\log dL)\cdot\log(ndL)}})$
NSN+Spectral	NSN	Spectral	$\frac{d}{p} = O(\frac{\log n}{\log(ndL)})$	-

Problem Formulation

- Subspaces are represented by their orthogonal bases $D_1, \ldots, D_L \in \mathbb{R}^{p \times d}$.
- Data points $y_1, \ldots, y_N$ have their own labels $w_1, \ldots, w_N \in [L]$ where y_i lies on subspace w_i.

$$y_i = D_{w_i} x_i, \quad x_i \sim \text{Unif}(\mathbb{S}^{d-1}), \quad \forall i \in [N].$$

- Task: Recover the labels $\{w_i : i \in [n]\}$.

Algorithms

- NSN+GSR : Run NSN to obtain neighborhood matrix $W \in \{0,1\}^{N \times N}$ and run GSR for it.
- NSN+Spectral : Run NSN to obtain W and run spectral clustering for $Z = W + W^\top$

Nearest Subspace Neighbor (NSN)

$y_i \leftarrow y_i / \|y_i\|_2, \forall i \in [N]$
for $i = 1, \ldots, N$ **do**
  $\mathcal{I}_i \leftarrow \{i\}$
  for $k = 1, \ldots, K$ **do**
    $\mathcal{U} \leftarrow \text{span}\{y_j : j \in \mathcal{I}_i\}$
    $j^* \leftarrow \arg\max_{j \in [N] \setminus \mathcal{I}_i} \|\text{Proj}_{\mathcal{U}} y_j\|_2$
    $\mathcal{I}_i \leftarrow \mathcal{I}_i \cup \{j^*\}$
  end for
  $W_{ij} \leftarrow \mathbb{I}_{j \in \mathcal{I}_i \text{ or } y_j \in \mathcal{U}}, \ \forall j \in [N]$
end for

Greedy Subspace Recovery (GSR)

$y_i \leftarrow y_i / \|y_i\|_2, \forall i \in [N]$
$\mathcal{W}_i \leftarrow \text{Top-}d\{y_j : W_{ij} = 1\}, \forall i \in [N]$
$\mathcal{I} \leftarrow [N], l \leftarrow 1$
while $\mathcal{I} \neq \emptyset$ **do**
  $i^* \leftarrow \arg\max_{i \in \mathcal{I}} \sum_{j=1}^N \mathbb{I}\{\|\text{Proj}_{\mathcal{W}_i} y_j\|_2 \geq 1 - \epsilon\}$
  $\hat{\mathcal{D}}_l \leftarrow \hat{\mathcal{W}}_{i^*}$
  $\mathcal{I} \leftarrow \mathcal{I} \setminus \{j : \|\text{Proj}_{\mathcal{W}_{i^*}} y_j\|_2 \geq 1 - \epsilon\}$
  $l \leftarrow l + 1$
end while
$\hat{w}_i \leftarrow \arg\max_{l \in [L]} \|\text{Proj}_{\hat{\mathcal{D}}_l} y_i\|_2, \ \forall i \in [N]$

Statistical Results

Fully random model - Subspaces are generated uniformly at random.

Theorem

Suppose n polynomial in d. There are constants $C_1, C_2 > 0$ such that if

$$\frac{n}{d} > C_1 \left(\log \frac{ne}{d\delta}\right)^2, \quad \frac{d}{p} < \frac{C_2 \log n}{\log(ndL\delta^{-1})},$$

then with probability $1 - \frac{3L\delta}{1-\delta}$, NSN+GSR and NSN+Spectral cluster the points exactly.

Semi-random model - Subspaces are fixed with $\max \text{aff} = \max_{i \neq j} \|D_i^\top D_j\|_F / \sqrt{d}$

Theorem

Suppose n polynomial in d. There are constants $C_1, C_2 > 0$ such that if

$$\frac{n}{d} > C_1 \left(\log \frac{ne}{d\delta}\right)^2, \quad \max \text{aff} < \sqrt{\frac{C_2 \log n}{\log(dL\delta^{-1}) \cdot \log(ndL\delta^{-1})}}.$$

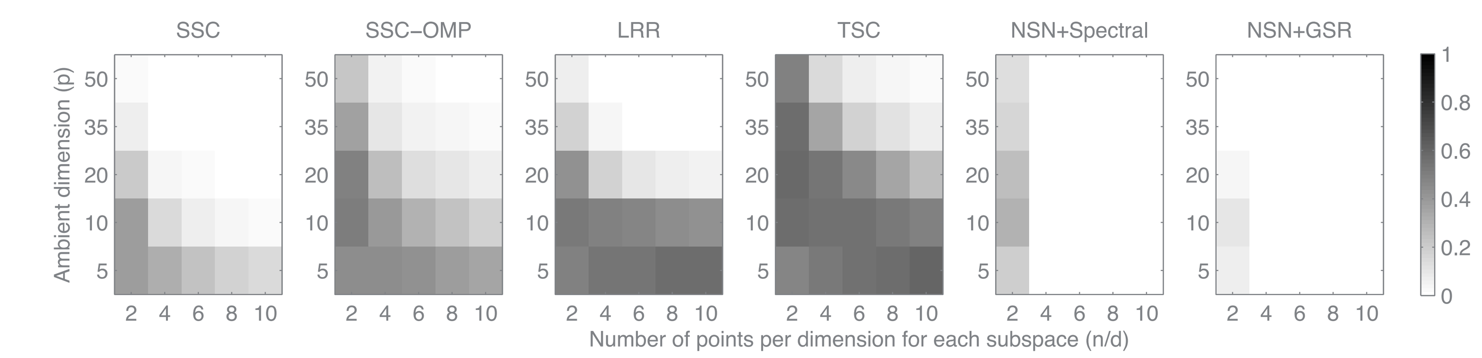
then with probability $1 - \frac{3L\delta}{1-\delta}$, NSN+GSR clusters the points exactly.

Implications

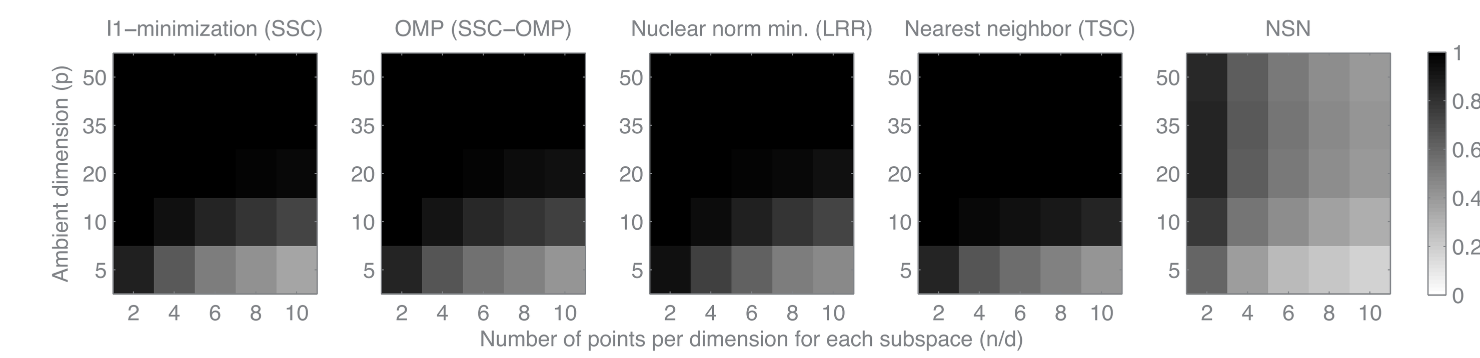
- One needs enough data points (linear in d)
- Subspaces should not be too close, but this condition improves as n grows.

Experimental Results - Synthetic Data

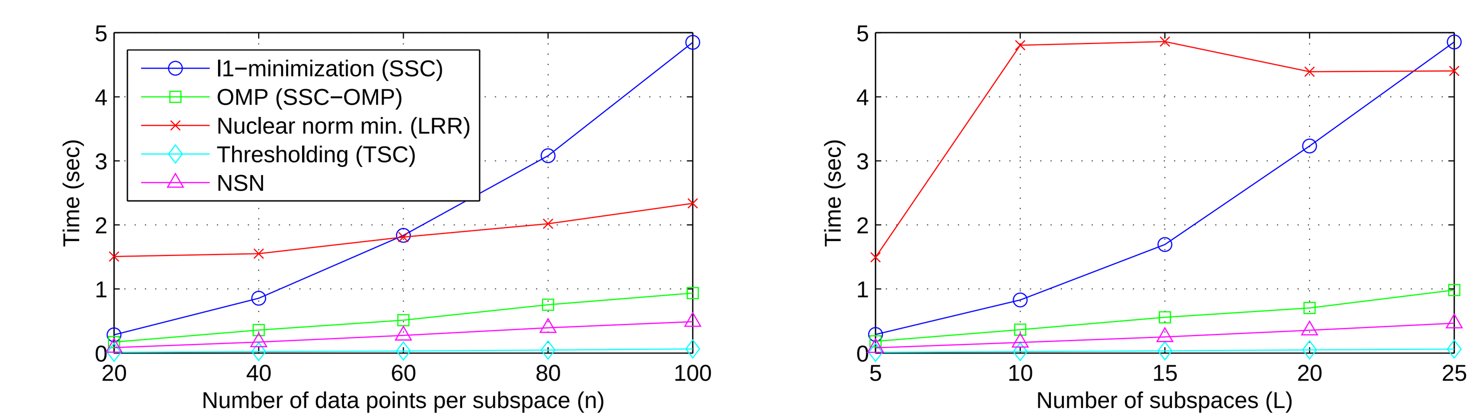
- Clustering Error in the fully random model ($d/p = 3/5$)



- Neighborhood Selection Error in the fully random model ($d/p = 3/5$)



- Average computational time for neighborhood construction



Experimental Results - Motion Segmentation

Hopkins 155 Dataset [7] - 155 video sequences of 2 or 3 motions.

L	Algorithms	K-means	K-flats	SSC	LRR	SCC	SSC-OMP	TSC	NSN+Spectral
2	Mean CE (%)	19.80	13.62	1.52	2.13	2.06	16.92	18.44	**3.62**
	Median CE (%)	17.92	10.65	0.00	0.00	0.00	12.77	16.92	**0.00**
	Avg. Time (sec)	-	0.80	3.03	3.42	1.28	0.50	0.50	**0.25**
3	Mean CE (%)	26.10	14.07	4.40	4.03	6.37	27.96	28.58	**8.28**
	Median CE (%)	20.48	14.18	0.56	1.43	0.21	30.98	29.67	**2.76**
	Avg. Time (sec)	-	1.89	5.39	4.05	2.16	0.82	1.15	**0.51**

Experimental Results - Face Clustering

Extended Yale B Dataset [8] - 64 images for each of L faces (48×42)

L	Algorithms	K-means	K-flats	SSC	SSC-OMP	TSC	NSN+Spectral
2	Mean CE (%)	45.98	37.62	1.77	4.45	11.84	**1.71**
	Median CE (%)	47.66	39.06	0.00	1.17	1.56	**0.78**
	Avg. Time (sec)	-	15.78	37.72	0.45	0.33	**0.78**
5	Mean CE (%)	73.77	55.51	4.79	8.93	11.90	**5.81**
	Median CE (%)	74.06	56.25	2.97	5.00	33.91	**4.69**
	Avg. Time (sec)	-	52.90	74.91	1.41	1.17	**5.62**
10	Mean CE (%)	82.68	62.72	9.43	15.32	39.48	**9.82**
	Median CE (%)	82.97	62.89	8.75	17.11	39.45	**9.06**
	Avg. Time (sec)	-	134.0	157.5	5.26	3.17	**14.73**

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